Probabilistic Linear Inversion of Reflection Seismic Data

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Outline

Motivation
Inverse Problems
Reflection Seismic Data
PhD Work

Work Areas
Quantification of Modeling Errors
Non-stationarity of the Prior Model
Noise Model Inference as Part of the Inversion Scheme

Conclusions
Overview
Outline

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Inverse Problems
Forward problem

▶ What is an *inverse* problem?
What is an inverse problem?

Let’s consider the opposite (i.e. the forward problem)

\[ d = g(m) \]  

Where \( d \) is data, \( m \) is model parameters, and \( g() \) is the relationship between \( m \) and \( d \).

In physics, \( g() \) is provided through physical models (laws).
What is an *inverse* problem?

Let’s consider the opposite (i.e. the *forward* problem)

\[ d = g(m) \]  \hspace{1cm} (1)

Where \( d \) is data, \( m \) is model parameters, and \( g() \) is the relationship between \( m \) and \( d \).

In physics, \( g() \) is provided through physical models (laws).

The inverse problem is the task of obtaining information about \( m \) given \( d \).
\[ d = g(m) \]

- Consider going to the doctor
  - \( m \): You have some sort of sickness/disease
  - \( g() \): Your body responds to this sickness
  - \( d \): We observe this response as symptoms
Inverse Problems

Example

\[ d_{\text{obs}} = g(m) + \epsilon \]  \hspace{1cm} (2)

- Problem: Symptoms could be explained by different sicknesses (non-unique solution)
  - Complex interplay within the body \((g())\) is non-linear
  - There might be symptoms that unrelated to sickness (errors: \(\epsilon\))
Inverse Problems

The subsurface

\[ d_{obs} = g(m) + \epsilon \] (3)

- **Problem:** Symptoms Geophysical data could be explained by different sicknesses Earth models (non-unique solution)
  - Complex interplay within the Earth \((g()\) is non-linear)
  - There might be noise sources that unrelated to the signal (errors: \(\epsilon\))
Problem: Many earth models could be explained by the same data response (non-unique solution)

Two approaches:
- Deterministic: Seek out the (a) model which best satisfies our data
  - Solved through minimization of misfit (optimization and regularization)
  - Ill-posed
- Probabilistic: Seek out an ensemble of probable models which satisfies the data
Probabilistic inverse problem → Full uncertainty characterization:

\[ \sigma_m(m) = k \rho_m(m)L(m) \] (4)

where \( \sigma_m(m) \) is the posterior probability, \( \rho_m(m) \) is the prior probability, \( L(m) \) is the likelihood function and \( k \) is a normalization constant.
Inverse Problems

Probabilistic solution

- Probabilistic inverse problem $\rightarrow$ Full uncertainty characterization:

$$\sigma_m(m) = k \rho_m(m) L(m)$$ (4)

- where $\sigma_m(m)$ is the \textit{posterior} probability, $\rho_m(m)$ is the \textit{prior} probability, $L(m)$ is the \textit{likelihood} function and $k$ is a normalization constant.

- In general, the likelihood function is given by:

$$L(m) = \int_D d \frac{\rho_d(d) \theta(d|m)}{\mu_D(d)}$$ (5)

- where $\rho_d(d)$ reflect measurement uncertainties, $\mu_D(d)$ is the homogeneous probability density, and $\theta(d|m)$ reflect modeling uncertainties.
The posterior probability density of the model parameters $\tilde{m}$ is described as a Gaussian probability distribution $\mathcal{N}(\tilde{m}, \tilde{C}_M)$ with mean:

$$\tilde{m} = \mu_M + (G C_M)^T C_D^{-1} (d_{\text{obs}} - G \mu_M)$$  \hspace{1cm} (6)$$

and covariance:

$$\tilde{C}_M = C_M - (G C_M)^T C_D^{-1} G C_M$$  \hspace{1cm} (7)$$

where $d_{\text{obs}}$ is the observed data, $C_M$ is the prior covariance, $\mu_M$ is the prior mean and $C_D$ is the data covariance.
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Reflection Seismic Data

Vessel
Reflection Seismic Data
Schematic Representation of Marine Acquisition
Reflection Seismic Data

Raw Data Example
Reflection Seismic Data

AVO/AVA Data

- Processed entity: Amplitude Versus Offset (AVO) or Amplitude Versus Angle (AVA)
Reflection Seismic Data

AVO Data Benefits

- Gives rise to the discipline of AVO analysis ("the search for bright/dim spots")
  - Zoeppritz Equations
- Use fast convolutional model as forward model in inversion
  - Linearizable
  - Computationally efficient!
- Data reduction
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  Overview
Title: Probabilistic Linear Inversion of Reflection Seismic Data

Manifested through three work areas

- Quantification of modeling errors (using linear forward models in a non-linear system)
- Non-stationarity of the prior model
- Noise model inference as part of the inversion scheme
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Conclusions
   Overview
Quantification of Modeling Errors

Problem

- Problem: Use imperfect forward model $g()$ when solving the inverse problem
- Approximative physics
  - Lack of understanding of the full problem (physics)
  - Lack of computational power
- Idea: Solve by generating a sample-based noise model which could potentially account for such errors
Consider one subsurface realization \( m \)

To generate one modeling error realization two ways of obtaining the forward response for \( m \) is compared:

- Linear (convolutional) model: \( d_{\text{conv}} = g_{\text{conv}}(m) = Gm \)
- Non-linear: \( d_{\text{nonl}} = g_{\text{nonl}}(m) \)

Modeling error: \( d_e = d_{\text{conv}} - d_{\text{nonl}} \)
Consider one subsurface realization $m$

To generate one modeling error realization two ways of obtaining the forward response for $m$ is compared:

- Linear (convolutional) model: $d_{\text{conv}} = g_{\text{conv}}(m) = Gm$
- Non-linear: $d_{\text{nonl}} = g_{\text{nonl}}(m)$

Modeling error: $d_e = d_{\text{conv}} - d_{\text{nonl}}$

Sample $\rightarrow \theta(d|m) \sim N\{d_{\text{Tapp}}, C_{\text{Tapp}}\}$
Quantification of Modeling Errors

Data Covariance

- Split data covariance in measurement uncertainty and "theory" (modeling) errors
  \[ C_D = C_d + C_T \]  
  \( (8) \)

- Use sample covariance as estimate of theory errors
  \[ C_D = C_d + C_{Tapp} \]  
  \( (9) \)
Quantification of Modeling Errors

Theory errors

- Linearized version of Zoeppritz equations
- Processing errors
  - Raw data $\rightarrow$ AVA data
Quantification of Modeling Errors

Prior Realizations

Realization from small-contrast model

- $v_p$ (m s$^{-1}$)
- $v_s$ (m s$^{-1}$)
- $\rho$ (kg m$^{-3}$)
- AI ($10^6$ Pa m$^{-3}$)
- $v_p/v_s$
Quantification of Modeling Errors

$C_{Tapp}$

$C_{Tapp1}$: Small-contrast model
Quantification of Modeling Errors

Inversion: Gaussian Prior
Quantification of Modeling Errors

Inversion: Gaussian Prior

AVO inversion (SN = 5): $\mathbf{C}_D = \mathbf{C}_d + \mathbf{C}_{\text{App1}}$
Quantification of Modeling Errors

Prior Realizations
Quantification of Modeling Errors

$C_{Tapp}$
Quantification of Modeling Errors

Inversion: Non-Gaussian Prior

AVO inversion (SN = 5): $C_D = C_d$
Quantification of Modeling Errors

Inversion: Non-Gaussian Prior

AVO inversion (SN = 5): $C_D = C_d + C_{Tapp1}$
Quantification of Modeling Errors

Inversion: Non-Gaussian Prior
Quantification of Modeling Errors

Conclusions

- Even lower limit modeling errors in seismic data are important to recognize and account for (Significant for S/N < 1)
- Normally modeling errors are ignored
- Novelty: Methodology allows for quantification and estimation of modeling errors
Quantification of Modeling Errors

Conclusions

- Even lower limit modeling errors in seismic data are important to recognize and account for (Significant for S/N < 1)
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- Novelty: Methodology allows for quantification and estimation of modeling errors
- Advantages:
  - Use 'cheap' (linear) forward, while acknowledging the more exact forward solution.
  - Independent of data
  - Gaussian description allows incorporation in linear-least squares solution
Even lower limit modeling errors in seismic data are important to recognize and account for (Significant for $S/N < 1$)

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Novelty: Methodology allows for quantification and estimation of modeling errors

Advantages:
- Use 'cheap' (linear) forward, while acknowledging the more exact forward solution.
- Independent of data
- Gaussian description allows incorporation in linear-least squares solution

Disadvantages:
- Reliant on prior model
- Lower resolution
Motivation

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Conclusions
Overview
Non-stationarity of the Prior Model

Problem

▶ Ullaberget, Svalbard, Arctic Ocean (Onshore equivalent to Barents Sea)
Non-stationarity of the Prior Model

Problem

- Statistical properties of physical parameters for rocks (acoustic impedance, density, porosity, resistivity etc.) can generally be considered to be non-stationary
Non-stationarity of the Prior Model

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Non-stationarity of the Prior Model

Problem

▶ Inversion: \( Posterior \propto Prior \times Likelihood \)

\[
\sigma_m(m) = k \rho_m(m)L(m) \tag{10}
\]

▶ Normally assume prior model with stationary variance for whole section
▶ Idea: Estimate variance of physical parameters prior to inversion using Bayesian inference
  ▶ Maximum likelihood estimator
  ▶ Sliding window technique
▶ Use this estimate as plug-in non-stationary variance of model parameters in prior model
Non-stationarity of the Prior Model

Base Case
Non-stationarity of the Prior Model

Inference of Variance
Non-stationarity of the Prior Model

Posterior Realizations
Non-stationarity of the Prior Model

Sensitivity
Non-stationarity of the Prior Model

Real World Case - NINI
Non-stationarity of the Prior Model

Real World Case - Inversion

![Images of seismic cross sections for different cases and models](https://example.com/images)
Non-stationarity of the Prior Model

Conclusions

- Novelty: Non-stationarity coupled to the variance of the physical parameter
Conclusions

- **Novelty:** Non-stationarity coupled to the variance of the physical parameter
- **Advantages:**
  - Non-stationarity taken into account
    - Realistic posterior realizations
    - More precise predictions in scenarios with heterogeneous variance in subsurface
- **Disadvantages:**
  - Dependency in the sources of information (Use data twice)
  - Loss of degrees of freedom
  - Loss of variance in results
  - Requires good knowledge of noise
Non-stationarity of the Prior Model

Conclusions

▶ Novelty: Non-stationarity coupled to the variance of the physical parameter

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- Overview
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Problem

▶ Inversion: \( \text{Posterior} \propto \text{Prior} \times \text{Likelihood} \)

\[
\sigma_m(m) = k \rho_m(m) L(m)
\]  

(11)

\[
L(m) = \int_D \mu_D(d) \theta(d|m)
\]  

(12)

▶ Correct evaluation of the likelihood is essential
  ▶ Requires a good model for the measurement and theory uncertainties
  ▶ Assume some noise model
  ▶ Infer from nearby or similar experiments, previous experiences, general considerations, measurement equipment uncertainty \( \rightarrow \) subjective choice of noise model

▶ Problem: Tend to involve human biases
Noise Model Inference as Part of the Inversion Scheme

Problem
Noise Model Inference as Part of the Inversion Scheme

Problem

[Diagrams showing data and power spectrum analysis]
Problem: Subjective choice of noise model

Idea: Proposed to let the data "decide" what is signal and what is noise

- Hierarchical Bayesian approach
- Assigning properties of the noise model with stochastic variables (hyperparameters $h$)
- Include these extra variables in the inversion scheme
Noise Model Inference as Part of the Inversion Scheme

Stochastic model

Figure: The stochastic model as a directed acyclic graph. The nodes represent stochastic variables and the black arrows show probability dependencies. The orange arrow between model parameters $\mathbf{m}$ and observed data $\mathbf{d}_{\text{obs}}$ indicates the deterministic relationship of the forward problem.
Noise Model Inference as Part of the Inversion Scheme

Noise models

1. A common practice for inferring noise is to assume an uncorrelated noise model:
   - Highest possible entropy (unpredictability)
   - Let $\mathbf{h} = h_1 = \sigma_d$
   - $\mathbf{C}_D = h_1^2 \mathbf{C}_d$

2. Use correct shape of noise model:
   - Let $\mathbf{h} = [h_1, h_2] = [\sigma_d, \sigma_T]$
   - $\mathbf{C}_D = h_1^2 \mathbf{C}_d + h_2^2 \mathbf{C}_T$

3. Use estimate/approximate shape of noise model:
   - Let $\mathbf{h} = [h_1, h_2] = [\sigma_d, \sigma_W]$
   - $\mathbf{C}_D = h_1^2 \mathbf{C}_d + h_2^2 \mathbf{C}_W$

   - Hyperprior: $h_1, h_2 \sim \text{Uniform}(0.00001, 1)$
Noise Model Inference as Part of the Inversion Scheme

Posterior $p(\mathbf{m}|\mathbf{d}_{\text{obs}})$: Case 1
Noise Model Inference as Part of the Inversion Scheme

Posterior $p(m|d_{obs})$: Case 2
Noise Model Inference as Part of the Inversion Scheme

Posterior $p(m|d_{obs})$: Case 3
Noise Model Inference as Part of the Inversion Scheme

Hyperposteriors $p(h|d_{obs})$
Noise Model Inference as Part of the Inversion Scheme

Comparison With Best Possible Linear Inversion
Noise Model Inference as Part of the Inversion Scheme

Conclusions

- **Novelty**: Critical review of inferring the noise model as part of the inversion
  - Considering colored noise
  - The hierarchical Bayes approach was in all cases able to accurately estimate the variance of the uncorrelated noise on the data
  - Case 1 shows noise being fitted as data when assuming uncorrelated noise in a scenario with correlated noise
  - Case 2 shows that it is possible to infer the variance of the noise if the correct shape is known
  - Case 3 shows that some improvements can be gained by assuming some correlation of error. Still see biased results!
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   Overview
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- Least-squares solutions continue to be the primary tool for large scale probabilistic seismic inversion
  - Computational demands
- Improvements on currently available methods:
  - Considering likelihood models and hence noise models
    - Correlated noise
    - Modeling errors
  - Non-stationarity in prior model
  - More in accordance with the expected heterogeneous appearance of the subsurface